Your Name

Your Signature

Student ID #

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. Make sure that it is easy for graders to follow what you are doing.
- Place a box around your answer to each question.
- **You may write on the backs of pages** (and are expected to for some questions, in order to have enough space). Both sides of each page of the exam will be scanned.
- Raise your hand if you have a question.
- This exam has 4 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	11	
2	11	
3	12	
4	16	
Total	50	

1. (11 points) **For this problem only, you do not have to show work or justification.** For each of the following statements, circle "T" to the left if the statement is true, and "F" if the statement is false. **Here "true" means "always true".** If the are both examples of and counterexamples to the statement, the correct answer is "false." If you don't know the answer almost immediately, just make a guess and move on; time is better spent on the other exam questions.

T F If $T(\mathbf{x}) = A\mathbf{x}$ is both injective and surjective, then A is invertible.

T F If A is not square then $T(\mathbf{x}) = A\mathbf{x}$ is not invertible.

T F If A is a 3×3 matrix and $A^2 = 0_{3\times 3}$, then $A = 0_{3\times 3}$ (here $0_{3\times 3}$ is the 3×3 matrix filled with all zeros.)

T F If $T : \mathbb{R}^m \to \mathbb{R}^n$ is surjective then $n > m$.

T F If $S \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and there is some $\mathbf{v} \in \mathbb{R}^n$ which is *not* in S, then $\dim S \leq n-1$.

T F If $S \subset \mathbb{R}^3$ is a subspace and $\mathbf{v}$ and $\mathbf{w}$ are two linearly independent vectors which are *not* in S, then $\dim S \leq 3-2 = 1$.

T F If A and B are 4×4 matrices and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^4$, then $(A+B)(\mathbf{v}+\mathbf{w}) = A\mathbf{v} + B\mathbf{w}$.

T F If $S \subset S' \subset \mathbb{R}^m$ are subspaces, then $\dim S \leq \dim S' \leq m$.

T F Suppose that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ spans a subspace S. Then $\dim S = 4$.

T F Suppose that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is a basis for S. Then every sublist of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is independent.

T F If $T : \mathbb{R}^m \to \mathbb{R}^n$ is not injective, then it is surjective.

2. (11 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Compute A^{-1}.

(b) Suppose

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solve the equation $AXA = A + BA$ for the matrix X. Give your final answer as an explicit matrix.

(c) Suppose that we know that

$$A\mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\mathbf{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A\mathbf{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find the vectors $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$. (Note: Your computation for this part should mostly be done already.)

3. (12 points) Consider the subspace

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5 : \begin{array}{c} x_1 + 2x_3 - x_4 + x_5 = 0 \\ x_2 - x_3 + x_4 = 0 \end{array} \right\}.$$

   (a) Give a basis for S. What is $\dim S$?

   (b) Find a list of vectors which spans S but is *not* a basis.

   (c) Find a matrix A with five rows such that $S = \text{row}(A)$.

   (d) Find an linear transformation T which is *not* injective (one-to-one) such that $S = \text{range}(T)$. Is T surjective (onto)?

   (e) Find a matrix B such that $S = \text{null}(B)$. What is the rank of B?

4. (16 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Consider the sub**set**

$$C = \{\mathbf{v} \in \mathbb{R}^2 : \|T(\mathbf{v})\| \leq \|\mathbf{v}\|\},$$

where here $\|\mathbf{v}\|$ is the length of the vector $\mathbf{v}$; i.e. $\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{x^2 + y^2}$. That is, C is the set of all vectors which are made shorter (or kept the same length) when input to T.*

(a) Write down the three conditions a subset $S \subset \mathbb{R}^2$ has to satisfy in order to be a sub**space**.

(b) Show that C satisfies *two* of the three conditions you wrote down in part (a). (You do not have to show whether or not it satisfies the remaining condition). Hint: the geometric interpretation of vector operations should help you here.

(c) Suppose (for this part and all following parts of the problem) that

$$T(\mathbf{x}) = \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}.$$

Note that then $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in C$ and $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin C$. Find another vector $\mathbf{w} \in C$ which is linearly independent from $\mathbf{e_1}$.

(d) Suppose that S is a subspace which contains C. What is the smallest possible dimension that S could have?† Explain. Give an example of such an S with this dimension.

(e) Suppose that S' is a subspace which is contained inside C. What is the largest possible dimension that S' could have? Explain. Give an example of such an S' with this dimension.

*You don't have to compute any square roots to do this problem. It might help to remember that whenever $a \leq b$, we know $\sqrt{a} \leq \sqrt{b}$.

†It might help you to draw a picture of what you think C looks like in order to do these last two parts.